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Momentum and angular momentum of two gravitating particles

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Abstract. In this paper, following the method developed by Bel, Salas and Sánchez Ron to solve the equations of predictive mechanics, and the method by Bel and Martin to obtain the momentum and angular momentum of a system of particles, we obtain the accelerations, momentum and angular momentum for two gravitating particles to the first order in G. As 'criterial' conditions for the determination of the acceleration we have used the Lorentz invariant equations obtained by Havas and Goldberg.

1. Introduction

In order to describe the motion of N point particles within the framework of general relativity, we must begin with the Einstein equations:

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=-8\pi GP_{\mu\nu}, \tag{1.1}$$

taking as tensor $P_{\mu\nu}$ the following expression:

$$P^{\mu\nu}=\sum_{i=1}^{N}\int_{-\infty}^{+\infty}p_i^{\mu\nu}(\tau_i)\delta^{(4)}(x-x_i(\tau_i))\,\mathrm{d}\tau_i. \tag{1.2}$$

The equations of motion are contained in the Einstein equations, since $P^{\mu\nu}_{;\nu}=0$, and from this it can be deduced that the particles must describe geodesics in relation to a metric solution of the field equations. The problem is that we need to know the motion of the particles in order to solve the field equations.

Methods of successive approximations have been used to solve the problem. The equations of motion of Einstein, Infeld and Hoffman are only valid for velocities small compared with the velocity of light. Havas and Goldberg (1962) developed a method in which each approximate equation of motion is Lorentz invariant and therefore valid for any velocity.

Here we shall merely summarise the results of the previously mentioned work of Havas. It is assured that the metric tensor $g_{\mu\nu}$ allows for a series expansion with the Minkowski metric as a zero-order approximation:

$$g_{\mu\nu}=\eta_{\mu\nu}+\sum_{1}^{\infty}{}_{n}g_{\mu\nu}. \tag{1.3}$$

$\eta_{\mu\nu}$ is taken with the signature $\{+1\ \ -1\ \ -1\ \ -1\}$.

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The world lines of each particle $x_i^\alpha(\tau_i)$ are parametrised with the proper time of special relativity given by:

$$d\tau_i = [\eta_{\alpha\beta}\, dx_i^\alpha\, dx_i^\beta]^{1/2}. \tag{1.4}$$

The derivatives with respect to τ will be represented by a point

$$u_i^\alpha = dx_i^\alpha/d\tau_i = \dot{x}_i^\alpha, \tag{1.5}$$

in such a way that

$$\eta_{\alpha\beta}u_i^\alpha u_i^\beta = 1, \qquad \eta_{\alpha\beta}u_i^\alpha \dot{u}_i^\beta = 0. \tag{1.6}$$

The following agreements will also be used:

$$(ab) = \eta_{\alpha\beta}a^\alpha b^\beta, \qquad a_\alpha = \eta_{\alpha\beta}a^\beta, \qquad \delta_{i\gamma} = \frac{\partial}{\partial x_i^\gamma}. \tag{1.7}$$

The laws of motion (Havas and Goldberg 1962) of the first and second order are respectively

$$(d/d\tau_i)\{{}_1M_i(u_{i\mu} + {}_1g_{\mu\rho}u_i^\rho) + {}_2M_i u_{i\mu}\} = \tfrac{1}{2}\ {}_1M_i \partial_{i\mu}\ {}_1g_{\rho\sigma}u_i^\rho u_i^\sigma, \tag{1.8}$$

$$\begin{aligned}(d/d\tau_i)[{}_1M_i(u_{i\mu} + {}_1g_{\mu\rho}u_i^\rho + {}_2g_{\mu\rho}u_i^\rho) + {}_2M_i(u_{i\mu} + {}_1g_{\mu\rho}u_i^\rho) + {}_3M_i u_{i\mu}]\\ = \tfrac{1}{2}[{}_1M_i\ \partial_{i\mu}\ {}_1g_{\rho\sigma}u_i^\rho u_i^\sigma + {}_2M_i\ \partial_{i\mu}\ {}_1g_{\rho\sigma}u_i^\rho u_i^\sigma + {}_1M_i\ \partial d_{i\mu}\ {}_2g_{\rho\sigma}u_i^\rho u_i^\sigma],\end{aligned} \tag{1.9}$$

where

$$\begin{aligned}&{}_1M_i = m_1(\text{constant}), \qquad {}_2M_i = -\tfrac{1}{2}m_i\ {}_1g_{\alpha\beta}u_i^\alpha u_i^\beta + {}_2C_i,\\ &{}_3M_i = -\tfrac{1}{2}m_i\ {}_2g_{\alpha\beta}u_i^\alpha u_i^\beta + \tfrac{3}{8}m_i({}_1g_{\mu\rho}u_i^\rho u_i^\mu)^2 + {}_3C_i.\end{aligned} \tag{1.10}$$

${}_2C_i$ and ${}_3C_i$ are constants of integration.

The equations of motion are obtained by substituting the solutions ${}_1g_{\mu\nu}$, ${}_2g_{\mu\nu}\ldots$ of the approximate field equations in the laws of motion. The constants of integration ${}_2C_i$ and ${}_3C_i$ are chosen in such a way that they do not appear infinite on the world lines of the particles. (In reality all this could only be achieved by the first order equations.)

The retarded and advanced solutions of the first order field equations,

$$\begin{aligned}&\Box\, {}_1\gamma_{\mu\nu} = -16\pi G \sum_{j=1}^{N} \int_{-\infty}^{+\infty} m_j u_{j\mu} u_{j\nu}\, \delta^{(4)}(x - x_j(\tau_j))\, d\tau_j,\\ &{}_1\gamma_{\mu\nu} = {}_1g_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\ {}_1g_{\alpha\beta}\end{aligned} \tag{1.11}$$

are

$${}_1g_{\mu\nu}^{(\epsilon)} = 4G\epsilon \sum_{j=1}^{N} \left(\frac{m_j(u_{j\mu}u_{j\nu} - \frac{1}{2}\eta_{\mu\nu})}{((x - x_j)u_j)}\right), \tag{1.12}$$

where $\epsilon = -1$ for the retarded solution and $+1$ for the advanced solution, and all the corresponding magnitudes for the particle (j), are calculated at the point of intersection of its world line with the past cone traced from x^μ if $\epsilon = -1$, or advanced if $\epsilon = +1$.

The time-symmetric solution is:

$${}_1g_{\mu\nu}^{(s)} = \tfrac{1}{2}({}_1g_{\mu\nu}^{(-1)} + {}_1g_{\mu\nu}^{(+1)}). \tag{1.13}$$

The equations of motion of the first order, retarded and advanced, contain similar terms to the radiation–reaction terms of electrodynamics. As it is not the purpose of this work to enter into a discussion of radiation phenomena we shall start from the time-symmetric equation of motion.

For an isolated system of two particles (i, j) the equations are

$$(\mathrm{d}/\mathrm{d}\tau_i)\{u_{i\mu} + {}^{i}_{1}g^{s}_{\mu\rho}u^{\rho}_{i} - \tfrac{1}{2}u_{i\mu}\,{}^{i}_{1}g^{s}_{\alpha\beta}u^{\alpha}_{i}u^{\beta}_{i}\} = \tfrac{1}{2}\partial_{i\mu}\,{}^{i}_{1}g^{s}_{\alpha\beta}u^{\alpha}_{i}u^{\beta}_{i}, \tag{1.14}$$

$${}^{i}_{1}g^{s}_{\mu\nu} = \tfrac{1}{2}({}^{i}_{1}g^{(-1)}_{\mu\nu} - {}^{i}_{1}g^{(+1)}_{\mu\nu}), \tag{1.15}$$

$${}^{i}_{1}g^{\epsilon}_{\mu\nu} = 4\pi\epsilon\eta_i m_j\left(\frac{u_{j\mu}u_{j\nu} - \frac{1}{2}\eta_{\mu\nu}}{(xu_j)}\right)_{\substack{x^2=0\\ \mathrm{sig}X^0=-\epsilon\eta_i}}, \tag{1.16}$$

with

$$x^{\mu} = x^{\mu}_{i} - x^{\mu}_{j}, \qquad \eta_i = +1, \qquad \eta_j = -1.$$

$\epsilon = -1$ for the retarded solution and $+1$ for the advanced solution.

In the expressions ${}^{i}_{1}g^{\epsilon}_{\mu\nu}$; $u_{j\mu}$ and (xu_j) they are calculated in the retarded position if $\epsilon = -1$ or in the advanced position if $\epsilon = +1$.

The equation of motion for the particle (j) is obtained by substituting i for j in (1.14) and (1.16).

In § 2, we shall see that the equation (1.14) can be put in the form

$$\mathrm{d}u_{i\mu}/\mathrm{d}\tau_i = \tfrac{1}{2}(W^{(-1)}_{i\mu}(\hat{x}, u_i, \hat{u}_j, \epsilon_i, \hat{\epsilon}_j) + W^{(+1)}_{i\mu}(\check{x}, u_i, \check{u}_j, \epsilon_i, \check{\epsilon}_j)). \tag{1.17}$$

$\hat{x}^{\mu}$, $\hat{u}^{\mu}_{j}$, $\hat{\epsilon}^{\mu}_{j}$ and $\check{x}^{\mu}$, $\check{u}^{\mu}_{j}$, $\check{\epsilon}^{\mu}_{j}$ are the relative position, 4-'velocity' and acceleration of (j) in the retarded and advanced positions respectively:

$$\hat{x}^{\mu} = x^{\mu}_{i}(\tau_i) - x^{\mu}_{j}(\tau_{jr}), \qquad \hat{u}^{\mu}_{j} = \left.\frac{\mathrm{d}x^{\mu}_{j}}{\mathrm{d}\tau_j}\right|_{\tau_j=\tau_{jr}}, \qquad \eta_{\alpha\beta}\hat{x}^{\alpha}\hat{x}^{\beta} = 0,$$

$$\hat{\epsilon}^{\mu}_{j} = \left.\frac{\mathrm{d}x^{\mu}_{j}}{\mathrm{d}\tau^{2}_{j}}\right|_{\tau_j=\tau_{jr}}, \qquad \mathrm{sig}[x^{0}_{i}(\tau_i) - x^{0}_{j}(\tau_{jr})] = \eta_i.$$

We note that the equations (1.17) are not independent. Contracting with u^{μ}_{i}, we obtain $u^{\mu}_{i}\,\mathrm{d}u_{i\mu}/\mathrm{d}\tau_i = 0$ and the functions $W^{(\epsilon)}_{i\mu}$ are verified as identities:

$$u^{\mu}_{i}W^{\epsilon}_{i\mu} = 0, \qquad \epsilon = \pm 1. \tag{1.18}$$

Combined with the fact that these equations are Lorentz invariants, this allows us to interpret them within the framework of predictive mechanics (Droz–Vincent 1970, Bel 1971).

Thus it can be supposed that the interaction can be described by a predictive Lorentz invariant system (PS)

$$\mathrm{d}u^{\mu}_{i}/\mathrm{d}\tau_i = \theta^{\mu}_{i}(x_i\ x_j\ u_i\ u_j), \tag{1.19}$$

where the accelerations must be a solution of the equations:

$$\theta^{\alpha}_{i}\frac{\partial\theta^{\mu}_{j}}{\partial u^{\alpha}_{i}} + u^{\alpha}_{i}\frac{\partial\theta^{\mu}_{j}}{\partial x^{\alpha}_{i}} = 0, \qquad \theta^{\alpha}_{i}u_{i\alpha} = 0, \qquad (1.20), (1.21)$$

and similar equations, obtained by permuting i and j.

The equations of motion (1.17) are considered as supplementary conditions in order to determine the accelerations, in such a way that every solution of the PS (1.19) satisfies the equations of motion (1.17).

Let us briefly recall how this is done: If $x_i^\alpha = \phi_i^\alpha(\tau_i, x, u)$ is a solution of the PS corresponding to the initial conditions $(x_i^\alpha\ x_j^\alpha\ u_i^\alpha\ u_j^\alpha)$, all the magnitudes $\hat{x}, \hat{u}_j, \hat{\epsilon}_j, \check{x}, \check{u}_j, \check{\epsilon}_j$ can be written as functions of the variables $(x_i^\alpha\ x_j^\beta\ u_i^\alpha\ u_j^\alpha)$. $\tilde{W}_{i\mu}^{(\epsilon)}(x, u, \theta_i, \theta_j)$ are the functions resulting from the substitution of these in $W_{i\mu}^{(\epsilon)}$.

The functions $\tilde{W}_{i\mu}$ satisfy the equations

$$\theta_j^\alpha \frac{\partial \tilde{W}_{i\mu}^\epsilon}{\partial u_j^\alpha} + u_j^\alpha \frac{\partial \tilde{W}_{i\mu}^\epsilon}{\partial x_j^\alpha} = 0, \qquad u_i^\mu \tilde{W}_{i\mu}^{(\epsilon)} = 0, \tag{1.22), (1.23}$$

and the supplementary conditions:

$$\tilde{W}_{i\mu}^{(\epsilon)}\Big|_{\substack{x^2=0\\ \mathrm{sig}\, x^0 = -\epsilon\eta_i}} = W_{i\mu}^{(\epsilon)}. \tag{1.24}$$

Therefore $\phi_i(\tau_i, x, u)$ will also be the solution of the equation of motion (1.17) if we choose θ_i^μ in such a way that

$$\theta_i^\mu = \tfrac{1}{2}[\tilde{W}_i^{(-1)\mu}(x, u, \theta_i, \theta_j) + \tilde{W}_i^{(+1)\mu}(x, u, \theta_i, \theta_j)]. \tag{1.25}$$

This equation can be solved within the framework of perturbation theory. If we write

$$\theta_i^\mu = \sum_1^\infty G^n\, {}^n\theta_i^\mu$$

and substitute in (1.25) we have

$${}^n\theta_i^\mu = \tfrac{1}{2}[{}^n\tilde{W}_i^{(-1)\mu}(x, u, {}^p\theta_i, {}^q\theta_j) + {}^n\tilde{W}_i^{(+1)\mu}(x, u, {}^p\theta_i, {}^q\theta_i)], \tag{1.26}$$

with $1 \leqslant p \leqslant n-1$ and $1 \leqslant q \leqslant n-1$.

The equation is then ready for a recurrence calculation. To the first order we have

$${}^1\theta_i^\mu = \tfrac{1}{2}[{}^1\tilde{W}_i^{(-1)\mu}(x, u) + {}^1\tilde{W}_i^{(+1)\mu}(x, u)]. \tag{1.27}$$

In this work in § 2, we only determine ${}^1\theta_i^\mu$ and we do not attempt to calculate the superior terms ${}^n\theta_i^\mu$, $n > 1$. We must note that only the series $\Sigma G^n\, {}^n\theta_i^\mu$ corresponds to an equivalent predictive system to the equation of motion of Havas. To see why we do this, and the significance of the approximate predictive system ${}^1\theta_i^\mu$, we see what will happen with the equations of motion of the superior order. Unfortunately, equations of the second order are as yet unobtainable. However, as the law of motion of the second order (1.9) differs from that of the first order (1.8) only in terms of the second order, it will be of the form

$$\frac{\mathrm{d}u_{i\mu}}{\mathrm{d}\tau_i} = \tfrac{1}{2}[W_{i\mu}^{(-1)} + W_{i\mu}^{(+1)}] + x_\mu(x, \hat{u}_j, \hat{\epsilon}_j, \check{u}_j, \check{\epsilon}_j, \ldots), \tag{1.28}$$

with $x_u = \mathrm{O}(G^2)$.

So the predictive system associated with this equation will be

$$\theta_i^\mu = G\, {}^1\theta_i^\mu + G^2\, {}^2\tilde{\theta}_i^\mu + \ldots, \tag{1.29}$$

where the first term coincides with that obtained from the equation of the first order.

Only the expressions ${}^n\theta_i^\mu$ for $n > 1$ will be different. Similarly, keeping in mind the laws of motion of superior order, the following could be concluded:

The PS corresponding to an equation of motion to the order N:

$$\theta_i^\alpha = G\, {}^1\theta_i^\alpha + G^2\, {}^2\theta_i^\alpha + \ldots + G^N\, {}^N\overline{\overline{\theta_i^\alpha}} \tag{1.30}$$

will have its $N-1$ first expressions identical to those obtained from the equations of motion of lower order.

In this way, at least formally, it is possible to determine a PS

$$\theta_i^\alpha = \sum G^n \, {}^n\theta_i^\alpha,$$

(where each ${}^n\theta_i^\alpha$ must be calculated from the equation of motion of the order N) which has solutions in common with the exact equation of motion

$$m_i \frac{\mathrm{d}}{\mathrm{d}\tau_i}\left[\frac{g_{\mu\rho}u_i^\rho}{(g_{\alpha\beta}u_i^\alpha u_i^\beta)^{1/2}}\right] = \frac{1}{2}\frac{m_i u_i^\rho u_i^\sigma}{(g_{\alpha\beta}u_i^\alpha u_i^\beta)^{1/2}}\partial_{i\mu}g_{\rho\sigma}, \tag{1.31}$$

where

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_1^\infty {}_n g_{\mu\nu}$$

is a solution of the field equation

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -8\pi G p_{\mu\nu}.$$

As a consequence, we interpret the PS of the first order calculated in § 2 as an approximated PS corresponding to the exact equation of motion; it is for this reason that we abandon the attempt to calculate ${}^2\theta_i^\alpha$, which must be done from an equation of motion of the second order.

2. An approximate predictive system for the two-body problem

In predictive mechanics an isolated system of two point particles (i, j) is described by a system of differential equations of the second order:

$$\frac{\mathrm{d}u_i^\alpha}{\mathrm{d}\tau_i} = \theta_i^\alpha(x_i \; x_j \; u_i \; u_j), \tag{2.1}$$

where x_i^α and x_j^α are the coordinates of the particles in Minkowski space. In order for the PS to be Lorentz invariant, we must express the accelerations θ_i^α in the form

$$\theta_i^\alpha = \eta_i a_i x^\alpha + b_{ii}u_i^\alpha + b_{ij}u_j^\alpha \tag{2.2}$$

with

$$x^\alpha = x_i^\alpha - x_j^\alpha; \qquad \eta_i = +1, \qquad \eta_j = -1.$$

a and b are functions of the invariants

$$\begin{aligned} x^2 &= \eta_{\alpha\beta}x^\alpha x^\beta, & (xu_i) &= \eta_{\alpha\beta}x^\alpha u_i^\beta, \\ (xu_j) &= \eta_{\alpha\beta}x^\alpha u_j^\beta, & (u_iu_j) &= \eta_{\alpha\beta}u_i^\alpha u_j^\beta. \end{aligned} \tag{2.3}$$

We take the metric tensor $\eta_{\alpha\beta}$ with the signature $(+1, -1, -1, -1)$. The accelerations must be the solutions of the equations

$$u_j^\rho \frac{\partial \theta_i^\alpha}{\partial x_j^\rho} + \theta_j^\rho \frac{\partial \theta_i^\alpha}{\partial u_j^\rho} = 0, \tag{2.4}$$

$$u_i^\rho \theta_{i\rho} = 0. \tag{2.5}$$

It is supposed that the accelerations can be developed in a power series of a coupling constant G:

$$\theta_i^\alpha = \sum_1^\infty G^n\, {}^n\theta_i^\alpha, \qquad {}^n\theta_i^\alpha = \eta_i^n a_i x^\alpha + {}^n b_{ii} u_i^\alpha + {}^n b_{ij} u_i^\alpha, \tag{2.6}$$

and substituting into (2.4) we have

$$D_j\, {}^n\theta_i^\alpha = {}^n A_i^\alpha, \qquad D_j = u_j^\rho \frac{\partial}{\partial x^\rho}, \tag{2.7}$$

where

$$ {}^1A_i^\alpha = 0, \tag{2.8}$$

with

$$ {}^n A_i^\alpha = \eta_i \sum_{p+q=n} {}^p\theta_j^\rho \frac{\partial\, {}^q\theta_i^\alpha}{\partial u_j^\rho}. \tag{2.9}$$

If we write

$$ {}^n A_i^\alpha = \eta_i^n A_i x_i^\alpha + {}^n B_{ii} u_i^\alpha + {}^n B_{ij} u_j^\alpha$$

the equations (2.4) and (2.5), supposing that they are satisfied order by order, give us

$$D_j\, {}^n a_i = {}^n A_i, \tag{2.10}$$

$$D_j\, {}^n b_{ii} = {}^n B_{ii}, \tag{2.11}$$

$$ {}^n b_{ij} = -\frac{(xu_i)\, {}^n a_i + {}^n b_{ii}}{(u_i u_j)}. \tag{2.12}$$

The general solution can be written in two different forms ($\epsilon = \pm 1$) (Bel *et al* 1973):

$$ {}^n a_i(\epsilon) = \int_{-\eta_i \epsilon r_j}^{(xu_j)} {}^n A_i\, \mathrm{d}(xu_j) + {}^n a_i^*(r_j\ s_j\ \kappa;\ \epsilon), \tag{2.13}$$

$$ {}^n b_{ii}(\epsilon) = \int_{-\eta_i \epsilon r_j}^{(xu_j)} {}^n B_{ii}\, \mathrm{d}(xu_j) + {}^n b_{ii}^*(r_j\ s_j\ \kappa; \epsilon), \tag{2.14}$$

$$ {}^n b_{ii}(\epsilon) = -\kappa^{-1}[{}^n a_i(e)(xu_i) + {}^n b_{ii}(\epsilon)], \tag{2.15}$$

where independent variables have been taken:

$$\kappa = (u_i u_j), \qquad s_j = (xu_i) - \kappa(xu_j), \tag{2.16}$$
$$r_j = [-x^2 + (xu_j)^2]^{1/2}(xu_j).$$

${}^n a_i^*(\epsilon)$ and ${}^n b_{ii}^*(\epsilon)$ are arbitrary functions of (κ, r_j, s_j).

Once the arbitrary functions have been chosen the general solution can be written in the form

$$ {}^n a_i \doteq \lambda\, {}^n a_i(-1) + (1-\lambda)\, {}^n a_i(+1). \tag{2.17}$$

In particular taking $\lambda = \frac{1}{2}$ we have the solution

$$ {}^n a_i = \tfrac{1}{2}({}^n a_i(-1) + {}^n a_i(+1)). \tag{2.18}$$

The following notation will be used

$$
\begin{aligned}
&{}^{n}a_i(\epsilon;\epsilon)={}^{n}a_i(\epsilon)\big|_{\mathrm{sig}\,x^0=-\epsilon\eta_i}^{x^2=0},\\
&{}^{n}a_i(\epsilon;-e)={}^{n}a_i(\epsilon)\big|_{\mathrm{sig}\,x^0=\epsilon\eta_i}^{x^2=0}.
\end{aligned}
\tag{2.19}
$$

Taking into account (2.13) and (2.14) we have

$$
\begin{aligned}
&{}^{n}a_i(\epsilon;\epsilon)={}^{n}a_i^{*}(\epsilon,\epsilon),\\
&{}^{n}b_{ii}(\epsilon;\epsilon)={}^{n}b_{ii}^{*}(\epsilon,\epsilon).
\end{aligned}
\tag{2.20}
$$

and remembering (2.8) we have

$$
\begin{aligned}
&{}^{1}a_i(\epsilon)={}^{1}a_i^{*}(\epsilon),\\
&{}^{1}b_{ii}(\epsilon)={}^{1}b_{ii}^{*}(\epsilon).
\end{aligned}
\tag{2.21}
$$

3. A first-order predictive system for two particles in gravitational interaction

The equation of motion (1.14) can be written in this way:

$$
\begin{aligned}
\frac{\mathrm{d}u_i^{\mu}}{\mathrm{d}\tau_i}=\frac{1}{2}\bigg[\bigg(&-\frac{\mathrm{d}g_{\mu\rho}^{(-1)}}{\mathrm{d}\tau_i}u_i^{\rho}-g_{\mu\rho}^{(-1)}\frac{\mathrm{d}u_i^{\rho}}{\mathrm{d}\tau_i}+\frac{1}{2}\frac{\mathrm{d}u_{i\mu}}{\mathrm{d}\tau_i}g_{\alpha\beta}^{(-1)}u_i^{\alpha}u_i^{\beta}\\
&+\frac{1}{2}u_{i\mu}\frac{\mathrm{d}g_{\alpha\beta}^{(-1)}}{\mathrm{d}\tau_i}u_i^{\alpha}u_i^{\beta}+u_{i\mu}g_{\alpha\beta}^{(-1)}u_i^{\alpha}\frac{\mathrm{d}u_i^{\beta}}{\mathrm{d}\tau_i}+\frac{1}{2}g_{\alpha\beta}^{(-1)}u_i^{\alpha}u_i^{\beta}\bigg)\\
&+(\ldots\text{similar terms with } g_{\alpha\beta}^{(+1)} \text{ instead of } g_{\alpha\beta}^{(-1)}\ldots)\bigg].
\end{aligned}
\tag{3.1}
$$

Taking into account the definitions of $\hat{x}_j$, $\hat{u}_j$, $\hat{\epsilon}_j$ we easily obtain, from the condition $x^2=0$, $\mathrm{sig}\,x^0=-\epsilon\eta_i$, that:

$$
\frac{\mathrm{d}\hat{u}_j}{\mathrm{d}\tau_i}=\frac{(\hat{x}u_i)}{(\hat{x}u_j)}\hat{\epsilon}_j,
\tag{3.2}
$$

$$
\frac{\mathrm{d}(\hat{x}\hat{u}_j)}{\mathrm{d}\tau_i}=\eta_i\bigg((u_i\hat{u}_j)-\frac{(\hat{x}u_i)}{(\hat{x}\hat{u}_j)}\bigg)+\frac{(\hat{x}u_i)}{(\hat{x}\hat{u}_j)}(\hat{x}\hat{\epsilon}_j),
\tag{3.3}
$$

with which can be calculated:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_i}({}_1^{i}g_{\mu\nu}^{\epsilon})=4Gm_j\epsilon\eta_i\bigg\{&\frac{(xu_i)}{(xu_j)^2}(u_{j\mu}\xi_{j\nu}+u_{j\nu}\xi_{j\mu})\\
&-\frac{u_{j\mu}u_{j\nu}-\frac{1}{2}\eta_{\mu\nu}}{(xu_j)^2}\bigg[\eta_i\bigg((u_iu_j)-\frac{(xu_i)}{(xu_j)}\bigg)+\frac{(xu_i)}{(xu_j)}(x\xi_j)\bigg]\bigg\},
\end{aligned}
\tag{3.4}
$$

$$
\partial_{i\gamma}\,{}_1^{i}g_{\mu\nu}^{\epsilon}=4Gm_j\epsilon\eta_i\bigg\{\frac{u_{j\mu}\xi_{j\nu}+u_{j\nu}\xi_{j\mu}}{(xu_j)^2}x_\gamma-\frac{u_{j\mu}u_{j\nu}-\frac{1}{2}\eta_{\mu\nu}}{(xu_j)^2}\bigg[\eta_i\bigg(u_{j\gamma}-\frac{x_\gamma}{(xu_j)}\bigg)+\frac{(x\xi_j)}{(xu_j)}x_\gamma\bigg]\bigg\},
\tag{3.5}
$$

where the magnitudes corresponding to the particles j are calculated in the retarded position if $\epsilon=-1$ or advanced position if $\epsilon=+1$. Substituting into the equation of motion (3.1), the latter can be put into the form:

$$
\frac{\mathrm{d}u_{i\mu}}{\mathrm{d}\tau_i}=\tfrac{1}{2}[W_{i\mu}^{(-1)}(\hat{x},u_i,\hat{u}_j,\xi_i,\hat{\xi}_j)+W_{i\mu}^{(+1)}(\check{x},u_i,\check{u}_j,\xi_j,\check{\xi}_j)].
\tag{3.6}
$$

Let us now determine a PS equivalent to this equation. If it is supposed that $\theta_i^\alpha = \Sigma G^n\, {}^n\theta_i^\alpha$ we have to first order in G:

$$ {}^1\theta_i^\alpha = \frac{1}{2}[{}^1\tilde{W}_i^{(-1)\alpha} + {}^1\tilde{W}_i^{(+1)\alpha}]. \tag{3.7} $$

${}^1\tilde{W}_{i\mu}^{(\epsilon)}$ must be the solution of the equations

$$ D_j\, {}^1\tilde{W}_{i\mu}^{(\epsilon)} = 0, \tag{3.8} $$

$$ u_i^\mu\, {}^1\tilde{W}_{u\mu} = 0, \tag{3.9} $$

and fulfil the supplementary condition:

$$ {}^1\tilde{W}_{i\mu}^{(\epsilon;\epsilon)} \equiv {}^1\tilde{W}_{i\mu}^{(\epsilon)}\big|_{\text{sig}x^0=-\epsilon\eta_i}^{x^2=0} = {}^1W_{i\mu}^{(\epsilon)}. \tag{3.10} $$

The expressions for ${}^1W_{i\mu}^{(\epsilon)}$, obtained directly from the equation of motion (3.1), are:

$$ {}^1W_{i\mu}^{(\epsilon)} = G\epsilon m_j\left[\frac{2(u_iu_j)^2-1}{(xu_j)^3}x_\mu - \frac{2(u_iu_j)^2+1}{(xu_j)^2}\left((u_iu_j) - \frac{(xu_i)}{(xu_j)}\right)u_{i\mu} + \frac{2(u_iu_j)^2-1}{(xu_j)^2}(xu_i)u_{j\mu}\right]. \tag{3.11} $$

If we write

$$ {}^1W_{i\mu}^{(\epsilon)} = \eta_i\, {}^1a_i^*(\epsilon,\epsilon)x_\mu + {}^1b_{ii}^*(\epsilon,\epsilon)u_{i\mu} + {}^1b_{ij}^*(\epsilon,\epsilon)u_{j\mu}, \tag{3.12} $$

we shall have

$$ {}^1a_i^*(\epsilon,\epsilon) = \eta_i\epsilon m_j\frac{2(u_iu_j)^2-1}{(xu_j)^2} \tag{3.13} $$

and

$$ {}^1b_{ii}^*(\epsilon,\epsilon) = -\epsilon m_j\frac{2(u_iu_j)^2+1}{(xu_j)^2}\left((u_iu_j) - \frac{(xu_i)}{(xu_j)}\right). \tag{3.14} $$

But the equations (2.16) and the conditions $x^2 = 0$, $\text{sig}\, x^0 = -\epsilon\eta_i$ give

$$ \begin{aligned} (xu_j) &= -\eta_i\epsilon r_j, \\ (xu_i) &= s_j - \eta_i\epsilon r_j\kappa, \end{aligned} \tag{3.15} $$

with which, substituting in (3.13) and (3.14), we have

$$ {}^1a_i^*(\epsilon,\epsilon) = -m_j\frac{2\kappa^2-1}{r_j^3}, \tag{3.16} $$

$$ {}^1b_{ii}^*(\epsilon,\epsilon) = -\eta_i m_j\frac{2\kappa^2+1}{r_j^3}s_j. \tag{3.17} $$

If we also write ${}^1\tilde{W}_{i\mu}^{(\epsilon)}$ as

$$ {}^1\tilde{W}_{i\mu}^{(\epsilon)} = \eta_i\, {}^1a_i(\epsilon)x_\mu + {}^1b_{ii}(\epsilon)u_{i\mu} + {}^1b_{ij}(\epsilon)u_{j\mu} \tag{3.18} $$

the solution of the equations (3.8) and (3.9) with the supplementary condition (3.10) is

$$ {}^1a_i(\epsilon) = -m_j\frac{2\kappa^2+1}{r_j^3}, \tag{3.19} $$

$$ {}^1b_{ii}(\epsilon) = -\eta_i m_j\frac{2\kappa^2+1}{r_j^3}s_j, \tag{3.20} $$

$$^{1}b_{ij}(\epsilon)=\eta_i m_j\frac{4\kappa s_j+(2\kappa^2-1)(xu_j)}{r_j^3}, \tag{3.21}$$

with which

$$^{1}\tilde{W}^{(\epsilon)}_{i\mu}=\eta_i m_j\left[-\frac{2\kappa^2-1}{r_j^3}x_\mu-\frac{2\kappa^2+1}{r_j^3}s_j u_{i\mu}+\frac{4\kappa s_j+(2\kappa^2-1)(xu_j)}{r_j^3}u_{j\mu}\right]. \tag{3.22}$$

We note that they do not depend on ϵ, that is

$$^{1}\tilde{W}^{(-1)}_{i\mu}={}^{1}\tilde{W}^{(+1)}_{i\mu}.$$

Substituting, finally, (3.22) in (3.7), we have the first-order predictive system:

$$^{1}\theta^{\mu}_i=\eta_i m_j\left[-\frac{2k^2-1}{r_j^3}x_\mu-\frac{2\kappa^2+1}{r_j^3}s_j u_{i\mu}+\frac{4\kappa s_j+(2\kappa^2-1)(xu_j)}{r_j^3}u_{j\mu}\right]. \tag{3.23}$$

4. Momentum and angular momentum to first order in *G*

Following Bel and Martin (1975), we shall determine the quadrivector momentum and the angular momentum, up to the first order in G, for two particles in gravitational interaction.

In the first place we change the signature of $\eta_{\mu\nu}$ in the equation (3.23). Therefore we now have:

$$\theta^{\mu}_i=\eta_i m_j\left[-\frac{2\kappa^2-1}{r_j^3}x^\mu+\frac{2\kappa^2+1}{r_j^3}s_j u^\mu_i-\frac{4\kappa s_j+(2\kappa^2-1)(xu_j)}{r_j^3}u^\mu_j\right], \tag{4.1}$$

and the formulae (2.16) will be

$$\begin{aligned}&\kappa=-(u_iu_j), \qquad (xu_j)=\eta_{\alpha\beta}x^\alpha u^\beta_j,\\ &r_j=(x^2+(xu_j)^2)^{1/2}, \qquad s_j=(xu_i)-\kappa(xu_j).\end{aligned} \tag{4.2}$$

With Bel and Martin we substitute into the system (2.1) the accelerations (4.1) for another equivalent system:

$$\pi^\alpha_i=\frac{dx^\alpha_i}{d\lambda_i}, \qquad \frac{d\pi^\alpha_i}{d\lambda_i}=\hat{\theta}^\alpha_i(x_i\ x_j\ \pi_i\ \pi_j), \tag{4.3}$$

where

$$\hat{\theta}^\alpha_i=\pi^2_i\theta^\alpha_i\left(x^\beta, u^\gamma_i\to\frac{\pi^\gamma_i}{\pi_i}, m_j\to\pi_j\right), \tag{4.4}$$

with

$$\pi^2_i=-(\pi_i\pi_i). \tag{4.5}$$

This system can be considered as the result of making the changes of parameter $\lambda_i=\tau_i/m_i$, $\lambda_j=\tau_j/m_j$ in (2.1).

To solve the system (4.3) we consider π^2_i, π^2_j as independent variables, and as $(\theta_i\pi_i)=0$ and $(\theta_j\pi_j)=0$, π^2_i, π^2_j are constants on the world lines of the particles. Identifying π^2_i and π^2_j with m^2_i and m^2_j we obtain the trajectories of the particles.

From the equation (4.4) and taking into account (4.1) we obtain:

$$ {}^1\hat{\theta}_i^\mu = \frac{\eta_i\pi_j\pi_i^2}{\hat{r}_j^3}\left\{-\left(\frac{2\hat{\kappa}^2}{\pi_i^2\pi_j^2}-1\right)x^\mu + \frac{2\hat{\kappa}^2+1}{\pi_i^2\pi_j^2}\hat{s}_j\frac{\pi_i^\mu}{\pi_i} - \left[\frac{4\hat{\kappa}\hat{s}_j}{\pi_i\pi_j} + \left(\frac{2\hat{\kappa}^2}{\pi_i^2\pi_j^2}-1\right)\frac{(x\pi_j)}{\pi_j}\right]\frac{\pi_j^\mu}{\pi_j}\right\}, \tag{4.6} $$

$$ \hat{r}_j = \left[x^2 + \frac{(x\pi_j)^2}{\pi_j^2}\right]^{1/2}, \qquad \hat{s}_j = \frac{1}{\pi_i\pi_j^2}[\pi_j^2(x\pi_i) - \hat{\kappa}(x\pi_j)] $$

$$ \hat{\kappa} = -(\pi_i\pi_j). $$

It is convenient to use a new set of independent variables $\{\hat{\kappa}, h^2, z_i, z_j, \pi_i^2, \pi_j^2\}$.

$$ \begin{aligned} &h^2 = \eta_{\alpha\beta}h^\alpha h^\beta, && h^\alpha = x^\alpha - z_i\pi_i^\alpha + z_j\pi_j^\alpha, \\ &z_i = \eta_i\Lambda^{-2}[\pi_j^2(x\pi_i) - \hat{\kappa}(x\pi_j)], && \Lambda^2 = \hat{\kappa}^2 - \pi_i^2\pi_j^2. \end{aligned} \tag{4.8} $$

From the definitions (4.8) we can obtain

$$ (x\pi_i) = \eta_i[-\pi_i^2 z_i + \hat{\kappa}z_j], \qquad x^2 = h^2 - z_i^2\pi_i^2 - z_j^2\pi_j^2 + 2\hat{\kappa}z_iz_j, \tag{4.9} $$

and taking into account (4.7–9) we have:

$$ \hat{s}_j = \frac{1}{\pi_i\pi_j^2}[\eta_i\pi_j^2(-\pi_1^2 z_i + \hat{\kappa}z_j) - \hat{\kappa}\eta_i(-\pi_j^2 z_j + \hat{\kappa}z_i)], \qquad \hat{r}_j = \left[h^2 + \frac{\Lambda^2}{\pi_j^2}z_i^2\right]^{1/2}. \tag{4.10} $$

It is also convenient to express the accelerations as a linear combination of the vectors:

$$ h^\alpha = x^\alpha - z_i\pi_i^\alpha + z_j\pi_j^\alpha, \qquad t_i^\alpha = \pi_j^2\pi_i^\alpha - \hat{\kappa}\pi_j^\alpha, \qquad t_j^\alpha = \pi_i^2\pi_j^\alpha - \hat{\kappa}\pi_i^\alpha, \tag{4.11} $$

which have the following properties:

$$ \begin{aligned} &(h\pi_i) = (h\pi_j) = 0, && (t_i\pi_j) = (t_j\pi_i) = 0, && (t_i\pi_i) = (t_j\pi_j) = \Lambda^2 \\ &\pi_i^\alpha\frac{\partial h^\beta}{\partial x_i^\alpha} = 0, && \pi_i^\alpha\frac{\partial z_j}{\partial x_i^\alpha} = 0, && \pi_i^\alpha\frac{\partial z_i}{\partial x_i^\alpha} = 0, \\ &\pi_i^\alpha\frac{\partial h^\beta}{\partial x_j^\alpha} = 0, && \pi_j^\alpha\frac{\partial z_j}{\partial x_j^\alpha} = 0, && \pi_j^\alpha\frac{\partial z_i}{\partial x_j^\alpha} = 0. \end{aligned} \tag{4.12} $$

The consequence of this is that if f is a function of

$$ \{\hat{\kappa}, h^2, z_i, z_j, \pi_i^2, \pi_j^2\}, $$

it can be verified that

$$ \pi_i^\alpha\frac{\partial f}{\partial x_i^\alpha} = \frac{\partial f}{\partial z_i}, \qquad \pi_j^\alpha\frac{\partial f}{\partial x_j^\alpha} = \frac{\partial f}{\partial z_j}. \tag{4.13} $$

If we write the equation (4.6) in the form

$$ \hat{\theta}_i^\alpha = \eta_i a_i x^\alpha + b_{ii}\pi_i^\alpha + b_{ij}\pi_j^\alpha \tag{4.14} $$

and if we make the change of base as indicated in (3.11) we easily obtain

$$ \begin{aligned} &\hat{\theta}_i^\alpha = \eta_i a_i h^\alpha + l_{ij}t_j^\alpha, \\ &l_{ij} = \frac{1}{\pi_i^2}(b_{ij} - a_i z_j). \end{aligned} \tag{4.15} $$

Taking into account (4.6) we obtain from (4.14) and (4.15):

$$\hat{\theta}_i^\alpha = -\eta_i \frac{\hat{\kappa}^2 + \Lambda^2}{\pi_j \hat{r}_j^3} h^\alpha + \frac{4\hat{\kappa}\pi_i^2\pi_j^2 + \hat{\kappa}\Lambda^2 - 3\hat{\kappa}^3}{\pi_i^2 \pi_j^3} \frac{z_i}{\hat{r}_j^3} t_j^\alpha. \tag{4.16}$$

The expression (4.16) is the starting point in obtaining the momentum and angular momentum, but before continuing we shall recall the definitions and methods of calculation:

Momentum must be an invariant vector under the Poincaré group satisfying

$$\pi_j^\alpha \frac{\partial P^\mu}{\partial x_i^\alpha} + \theta_i^\alpha \frac{\partial P^\mu}{\partial \pi_i^\alpha} = 0. \tag{4.17}$$

The angular momentum must be a second order tensor, skew-symmetric, invariant under the Lorentz group and of the form

$$\pi_i^\alpha \frac{\partial J^{\mu\nu}}{\partial x_i^\alpha} + \theta_i^\alpha \frac{\partial J^{\mu\nu}}{\partial \pi_i^\alpha} = 0, \tag{4.18}$$

$$\frac{\partial J^{\alpha\beta}}{\partial x_i^\gamma} + \frac{\partial J^{\alpha\beta}}{\partial x_j^\gamma} = \delta_\gamma^\beta p^\alpha - \delta_\gamma^\alpha p^\pi. \tag{4.19}$$

Furthermore the following asymptotic conditions are imposed:

$$\lim_{x^2 \to \infty_P} P^\mu = P_0^\mu, \tag{4.20}$$

$$\lim_{x^2 \to \infty_P} J^{\mu\nu} = J_0^{\mu\nu}, \tag{4.21}$$

where P_0^μ and $J_0^{\mu\nu}$ are the momentum and the angular momentum of two free particles. That is to say P^μ and $J^{\mu\nu}$ must coincide in the infinite past with the corresponding expressions for free particles.

If f is a function of the variables $\{\hat{\kappa}, h^2, z_i, z_j, \pi_i^2, \pi_j^2\}$, we define the infinite past limit, $\lim_{x^2 \to \infty_P} f = f_0$ as follows:

$$\text{(a)} \quad \lim_{\substack{z_i \to -\infty \\ z_j \to -\infty \\ \forall h^2}} f = f_0, \tag{4.22}$$

$$\text{(b)} \quad \lim_{\substack{h^2 \to \infty \\ \forall z_i, z_j}} f = f_0. \tag{4.23}$$

The method of solving the equations (4.17, 18, 19) is based on the following result:

If some functions of the phase-space $\{q_i^\alpha \; q_i^\beta \; p_i^\alpha \; p_j^\beta\}$ are chosen, such as $q_i^\alpha - x_i^\alpha$ and p_i^α, which are invariant vectors under the Poincaré group and solutions of the equations

$$\pi_i^\alpha \frac{\partial p_j^\mu}{\partial x_i^\alpha} + \theta_i^\alpha \frac{\partial p_j^\mu}{\partial u_i^\alpha} = 0, \tag{4.24}$$

$$\pi_i^\alpha \frac{\partial p_i^\mu}{\partial x_i^\alpha} + \theta_i^\alpha \frac{\partial q_i^\mu}{\partial \pi_i^\alpha} = p_i^\mu, \tag{4.25}$$

$$\pi_i^\alpha \frac{\partial q_j^\mu}{\partial x_i^\alpha} + \theta_i^\alpha \frac{\partial q_j^\mu}{\partial \pi_i^\alpha} = 0, \tag{4.26}$$

(and the similar equations obtained by permuting i and j) then the following expressions are solutions of the equations (4.17, 18, 19):

$$P^\mu = p_i^\mu + p_j^\mu, \tag{4.27}$$

$$J^{\mu\nu} = q_i^\mu p_i^\nu - q_i^\nu p_i^\mu + q_j^\mu p_j^\nu - q_j^\nu p_j^\mu, \tag{4.28}$$

and if furthermore we require the asymptotic conditions:

$$\lim_{x^2 \to \infty_P} q_i^\mu = x_i^\mu, \tag{4.29}$$

$$\lim_{x^2 \to \infty_P} p_i^\mu = \pi_i^\mu, \tag{4.30}$$

then (4.27) and (3.28) have the asymptotic behaviour described in (3.20, 21). On the other hand, if P_i^μ is a solution of (4.24) and satisfies the asymptotic condition (4.30) it fulfils

$$(p_i p_i) = (\pi_i \pi_i). \tag{4.31}$$

The equations (4.24, 25, 26) are solved in the framework of perturbation theory. To this end we substitute in these equations the following expressions:

$$p_i^\beta = \pi_i^\beta + G[\eta_i\, {}^1\alpha_i h^\beta + {}^1\mu_{ij} t_j^\beta], \tag{4.32}$$

$$q_i^\beta = x_i^\beta + G[\eta_i\, {}^1\gamma_i h^\beta + {}^1\nu_{ii} t_i^\beta + {}^1\nu_{ij} t_j^\beta]. \tag{4.33}$$

(in (4.32) no parallel vector to t_i^β appears due to the condition (4.31) being satisfied.) By substituting (4.32) in the equation (4.24) the following equations are obtained: (We resolve order by order and take into account the properties (4.13)):

$$\frac{\partial\, {}^1\alpha_i}{\partial z_i} = -{}^1a_i, \qquad \frac{\partial\, {}^1\alpha_i}{\partial z_j} = 0, \tag{4.34}$$

$$\frac{\partial\, {}^1\mu_{ij}}{\partial z_i} = -{}^1l_{ij}, \qquad \frac{\partial\, {}^1\mu_{ij}}{\partial z_j} = 0. \tag{4.35}$$

Taking into account (4.15, 16) we obtain the expressions for ${}^1a_i$ and ${}^1l_{ij}$ into which the previous equations are converted

$$\frac{\partial\, {}^1\alpha_i}{\partial z_i} = \frac{\hat\kappa^2 + \Lambda^2}{\pi_j \hat r_j^3}, \qquad \frac{\partial\, {}^1\alpha_i}{\partial z_j} = 0, \tag{4.36}$$

$$\frac{\partial\, {}^1\mu_{ij}}{\partial z_i} = -\frac{4\hat\kappa\pi_i^2\pi_j^2 + \hat\kappa\zeta^2 - 3\hat\kappa^2}{\pi_i^2\pi_j^3}\frac{z_i}{\hat r_j^3}: \qquad \frac{\partial\, {}^1\mu_{ij}}{\partial z_j} = 0. \tag{4.37}$$

The solution of (4.36) which satisfies the asymptotic conditions (4.30) is

$${}^1\alpha_i = \frac{\hat\kappa^2 + \Lambda^2}{\pi_j}\int_{-\infty}^{z_i} \frac{\mathrm{d}z_i}{\hat r_j^3} = \frac{\hat\kappa^2 + \Lambda^2}{h^2}\left[\frac{z_i}{\pi_j \hat r_j} + \frac{1}{\Lambda}\right], \tag{4.38}$$

$${}^1\mu_{ij} = -\frac{4\hat\kappa\pi_i^2\pi_j^2 + \hat\kappa\Lambda^2 - 3\hat\kappa^3}{\pi_i^2\pi_j^3}\int_{-\infty}^{z_i} \frac{z_i\,\mathrm{d}z_i}{\hat r_j^3} = \frac{\hat\kappa^3 - 3\hat\kappa\Lambda^2}{\Lambda^2\pi_i^2\pi_j\hat r_j}. \tag{4.39}$$

Substituting (4.38, 39) in (4.32), and taking into account (4.27) we obtain the quadrivector momentum:

$$P^\alpha = \pi_i^\alpha + \pi_j^\alpha + G\left[\left(\frac{\hat\kappa^2 + \Lambda^2}{h^2\pi_j\hat r_j}z_i - \frac{\hat\kappa^2 + \Lambda^2}{h^2\pi_i\hat r_i}z_j\right)h^\alpha + \frac{\hat\kappa^3 - 3\hat\kappa\Lambda^2}{\Lambda^2\pi_i^2\pi_j\hat r_j}t_j^\alpha + \frac{\hat\kappa^3 - 3\kappa\Lambda^2}{\Lambda^2\pi_j^2\pi_i\hat r_i}t_i^\alpha\right]. \tag{4.40}$$

The functions $q^{\alpha}_{i,j}$ are similarly determined. On substituting (4.33) in (4.25, 6) we obtain:

$$\frac{\partial^1 \gamma_i}{\partial z_i} = {}^1\alpha_i, \qquad \frac{\partial^1 \gamma_i}{\partial z_j} = 0, \tag{4.41}$$

$$\frac{\partial^1 \nu_{ij}}{\partial z_i} = {}^1\mu_{ij}, \qquad \frac{\partial^1 \nu_{ij}}{\partial z_j} = 0, \tag{4.42}$$

$$\frac{\partial^1 \nu_{ii}}{\partial z_i} = 0, \qquad \frac{\partial^1 \nu_{ii}}{\partial z_j} = 0. \tag{4.43}$$

As in the case of the electromagnetic interaction (Bel and Martin 1975) the previous equations do not allow for a solution with asymptotic behaviour (4.29), however, there exist solutions which satisfy a weaker condition:

$$\lim_{x^2 \to \infty_P} x^{-1}(q^{\alpha}_i - x^{\alpha}_i) = 0. \tag{4.44}$$

The solutions of (4.41, 42, 43) which fulfil conditions (4.44) are the following:

$${}^1\gamma_i = \frac{\hat{\kappa}^2 + \Lambda^2}{h^2\Lambda^2}[\pi_j \hat{r}_j + z_i \Lambda], \tag{4.45}$$

$${}^1\nu_{ij} = \frac{\hat{\kappa}^3 - 3\hat{\kappa}\Lambda^2}{\Lambda^3 \pi_i^2} \ln\left[z_i + \left(z_i^2 + \frac{h^2\pi_j^2}{\Lambda^2}\right)^{1/2}\right] + \nu^*_{ij}, \tag{4.46}$$

$${}^1\nu_{ii} = \nu^*_{ii}, \tag{4.47}$$

where ν^*_{ij}, ν^*_{ii} are functions of h^2, $\hat{\kappa}$, π_i^2, π_j^2 such that

$$\lim_{x^2 \to \infty_P} h^{-1}\nu^*_{ij} = 0, \qquad \lim_{x^2 \to \infty_P} h^{-1}\nu^*_{ii} = 0. \tag{4.48}$$

Substituting in (4.33) the expressions (4.45, 46, 47) we obtain:

$$q^{\beta}_i = x^{\beta}_i + G\left\{\eta_i\left(\frac{\hat{\kappa}^2 + \Lambda^2}{h^2\Lambda^2}\pi_j \hat{r}_j + \frac{\hat{\kappa}^2 + \Lambda^2}{\Lambda h^2} z_i\right)h^{\beta} + \nu^*_{ii} t^{\beta}_i + \nu^*_{ij} t^{\beta}_j \right.$$
$$\left. + \frac{\hat{\kappa}^3 - 3\hat{\kappa}\Lambda^2}{\Lambda^3\pi_i^2} \ln\left[z_i + \left(z_i^2 + \frac{h^2\pi_j^2}{\Lambda^2}\right)^{1/2}\right] t^{\beta}_j\right\}. \tag{4.49}$$

Substituting (4.49) in (4.28) the angular momentum is obtained, but we shall not give its explicit expression which depends on the arbitrary functions ν^*_{ij}, ν^*_{ii}. However, we shall give the intrinsic angular momentum expression, which does not depend on any arbitrary function.

The intrinsic angular momentum is defined by

$$W^{\alpha} = \tfrac{1}{2} P^{-1} \delta^{\alpha\beta\lambda\mu} P_{\beta} J_{\lambda\mu},$$
$$P = [-(PP)]^{1/2}; \qquad \delta^{0123} = +1. \tag{4.50}$$

Substituting (4.27, 28, 32, 33) into (4.50) and maintaining only first-order terms, we obtain

$$W^{\alpha} = W^{\alpha}_0 + G\,{}^1W\eta^{\alpha}, \qquad \eta^{\alpha} = \delta^{\alpha\beta\lambda\mu} x_{\beta} \pi_{i\lambda} \pi_{j\mu}, \tag{4.51}$$

$${}^1W = M^{-1}[{}^1\gamma_i + {}^1\gamma_j - z_i\,{}^1\alpha_i - z_j\,{}^1\alpha_j + (\Lambda^2 M^{-2} - \hat{\kappa})({}^1\mu_{ij} + {}^1\mu_{ji})],$$

$$M^2 = \pi_i^2 + \pi_j^2 + 2\hat{\kappa},$$

and taking (4.38, 39, 45) into account we have

$$^1W = M^{-1}\left[\frac{\hat{\kappa}^2+\Lambda^2}{h^2\Lambda^2}(\pi_j\hat{r}_j+\pi_i\hat{r}_i)-\frac{\hat{\kappa}^2+\Lambda^2}{h^2}\left(\frac{z_i^2}{\pi_j\hat{r}_j}+\frac{z_j^2}{\pi_i\hat{r}_i}\right)\right.$$
$$\left.+(\Lambda^2M^{-2}-\hat{\kappa})\left(\frac{1}{\pi_i^2\pi_j\hat{r}_j}+\frac{1}{\pi_j^2\pi_i\hat{r}_i}\right)\frac{\hat{\kappa}^3-3\hat{\kappa}\Lambda^2}{\Lambda^2}\right]. \tag{4.52}$$

5. Gravitational energy and 3-accelerations

In order to interpret the results of the previous section it is convenient to consider the tridimensional predictive system associated with every invariant PS (Bel and Martin 1975).

The Hamiltonian, and energy, of the system are defined as:

$$H = -\bar{P}_0, \tag{5.1}$$

where $\bar{P}_0$ means that in the expression for P_0 given in (4.40) the following substitutions must be made:

$$x_i^0 = x_j^0 = t, \qquad \pi_i^2 = m_i^2,$$
$$\pi_i^0 = \frac{m_i}{(1-v_i^2)^{1/2}}, \qquad \pi_i^s = \frac{m_i v_i^s}{(1-v_i^2)^{1/2}}, \qquad s = 1, 2, 3 \tag{5.2}$$
$$v_i^s = \frac{\mathrm{d}x_i^s}{\mathrm{d}t}.$$

We calculate the energy in the rest frame of the particles j:

$$H = -\frac{(\overline{\pi_j P})}{m_j}, \tag{5.3}$$

and taking into account (4.40, 1, 2) and (5.2) we obtain

$$H = \pi_i^0 + m_j - G\frac{m_j\pi_i^0(3m_i^2-2\pi_i^{02})}{m_i^2|\boldsymbol{x}|}, \tag{5.4}$$
$$|\boldsymbol{x}| = ((x^1)^2+(x^2)^2+(x^3)^2)^{1/2}.$$

If the particle i is also at rest this expression reduces to

$$H = m_i + m_j - G\frac{m_i m_j}{|\boldsymbol{x}|} \tag{5.5}$$

which coincides with the gravitational energy in Newtonian mechanics.

It is easy to see that the energy of interaction $(H - \pi_i^0 - m_j)$ is negative for

$$\pi_i^0 < \pi_{ic}^0,$$

zero for

$$\pi_i^0 = \pi_{ic}^0,$$

and positive for

$$\pi_i^0 > \pi_{ic}^0,$$

with $\pi_{ic}^0 = (\frac{3}{2})^{1/2} m_i$, corresponding to a speed $v_i = 0{\cdot}577c$.

The expression (5.4) must coincide in some way with the energy of a particle in the constant gravitational field created by a body with spherical symmetry.

The energy of a particle in a constant gravitational field is given by:

$$\xi = P_0 = m_j g_{00}(\mathrm{d}x^0/\mathrm{d}s). \tag{5.6}$$

We substitute in (5.6) the Schwarzschild metric in harmonic coordinates (Fock 1959, Hirondel 1974):

$$\mathrm{d}s^2 = \frac{r-r_0}{r+r_0}\mathrm{d}t^2 - \frac{r+r_0}{r-r_0}\mathrm{d}r^2 - (r+r_0)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{5.7}$$

$$r_0 = Gm_j.$$

From (5.7) we obtain

$$g_{00} = \frac{r-r_0}{r+r_0}, \qquad \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{r-r_0}{r+r_0} - \frac{r+r_0}{r-r_0}\dot{r}^2 - (r+r_0)^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{5.8}$$

where $\dot{r} = \mathrm{d}r/\mathrm{d}t$, $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$, $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$, $t = x^0$. Expanding (5.8) in powers of r_0 and retaining only terms to first order in r_0 we have:

$$\begin{aligned} &g_{00} = 1 - 2r_0/r,\\ &(\mathrm{d}s/\mathrm{d}t)^2 = 1 - \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - 2(r_0/r)[1 + \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)]. \end{aligned} \tag{5.9}$$

Making the change of coordinates

$$\begin{aligned} &x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,\\ &z = r\cos\theta, \qquad t = t. \end{aligned} \tag{5.10}$$

(5.9) can be written in the form:

$$\begin{aligned} &g_{00} = 1 - 2r_0/r,\\ &(\mathrm{d}s/\mathrm{d}t)^2 = 1 - v^2 - 2(r_0/r)(1+v^2),\\ &v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,\\ &r^2 = x^2 + y^2 + z^2. \end{aligned} \tag{5.11}$$

From (5.11) we easily obtain

$$\mathrm{d}t/\mathrm{d}s = (1-v^2)^{1/2}\left(1 + \frac{r_0}{r}\frac{1+v^2}{1-v^2}\right). \tag{5.12}$$

Now substituting (5.12) and the expression of g_{00}, given in (5.11), in (5.6) we have:

$$\xi = \frac{m_i}{(1-v_i^2)^{1/2}} + \frac{r_0}{r}m_i\left(\frac{-1+3v_i^2}{(1-v_i^2)^{3/2}}\right), \tag{5.13}$$

and if we put $\pi_i^0 = m_i/(1-v_i^2)^{1/2}$ (5.13) can be written in the form

$$\xi = \pi_i^0 + \frac{r_0}{r}\pi_i^0\left(-3 + 2\frac{(\pi_i^0)^2}{m_i^2}\right), \tag{5.14}$$

which coincides with the expression (5.4), giving the energy of the system if we add the rest energy of the particle which creates the field.

Finally from ${}^1\theta_i^\alpha$ given by (4.1) we can get the 3-accelerations up to the first order in G, having in mind the relation

$$a_i^\kappa = (1-v_i^2)({}^1\theta_i^\kappa|_{t_i=t_j} - {}^1\theta_i^0|_{t_i=t_j} v_i^\kappa), \qquad \kappa = 1, 2, 3, \tag{5.15}$$

$$a_i^\kappa = \mathrm{d}^2 x_i^\kappa/\mathrm{d}t.$$

If (5.15) is expanded in powers of (v/c), retaining only the second-order terms we obtain

$$a_i^\kappa = -\eta_i \frac{Gm_j}{|\boldsymbol{x}|^3}\left[1 - \frac{3}{2}\frac{(\boldsymbol{x}\cdot\boldsymbol{v}_j)^2}{|\boldsymbol{x}|^2} - 4(\boldsymbol{v}_i\cdot\boldsymbol{v}_j) + 2v_j^2 + v_i^2\right]x^\kappa$$

$$- \eta_i \frac{Gm_j}{|\boldsymbol{x}|^3}[4(\boldsymbol{x}\cdot\boldsymbol{v}_i) - 3(\boldsymbol{x}\cdot\boldsymbol{v}_j)](v_j^\kappa - v_i^\kappa). \tag{5.16}$$

These accelerations coincide with those given by Einstein, Infeld and Hoffman, except the second-order term in G:

$$\frac{G^2 m_i m_j (5m_i + 4m_j)}{|\boldsymbol{x}|^4}\boldsymbol{x}. \tag{5.17}$$

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